Math 10A
Worksheet, Discussion #5; Friday, 6/22/2018
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1 Derivative Definition

1.1 Concepts

1. The **derivative** of a function f at x_0 can be written as

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

1.2 Example

2. Find $\lim_{x \to 1} \frac{e^{3x} - e^3}{x^2 - 1}$.

Solution: We can factor the bottom as $(x-1)(x+1)$. Letting $f(x) = e^{3x}$, we recognize this derivative as

$$\lim_{x \to 1} \frac{e^{3x} - e^3}{x^2 - 1} = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \cdot \frac{1}{x+1} = \frac{f'(1)}{2} = \frac{3e}{2}.$$

1.3 Problems

3. Find $\lim_{x \to 1} \frac{e^{\sqrt{x}} - e}{x^2 - 3x + 2}$.

Solution: Let $f(x) = e^{\sqrt{x}}$ so that $f'(x) = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$. Then we can factor the bottom as $(x-1)(x-2)$ and the limit is

$$\lim_{x \to 1} \frac{e^{\sqrt{x}} - e}{x^2 - 3x + 2} = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \cdot \frac{1}{x-2} = \frac{f'(1)}{-1} = \frac{-e}{2}.$$

4. Find $\lim_{x \to 0} \frac{\cos x - 1}{x^2 + x}$.

Solution: Let $f(x) = \cos x$. Then the limit is

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2 + x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \cdot \frac{1}{x+1} = \frac{f'(0)}{1} = 0.$$

5. Find $\lim_{x\to\pi} \frac{\sin x}{x - \pi}$.

Solution: Let $f(x) = \sin x$. Then $f(\pi) = \sin \pi = 0$. So

$$\lim_{x\to\pi} \frac{\sin x}{x - \pi} = \lim_{x\to\pi} \frac{f(x) - f(\pi)}{x - \pi} = f'(\pi) = \cos(\pi) = -1.$$

6. Find $\lim_{x\to 0} \frac{\tan x}{x}$.

Solution: Let $f(x) = \tan x$. Then $f(0) = 0$ and so

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = \sec^2(0) = 1.$$

1.4 Extra Problems

7. Find $\lim_{x\to 0} \frac{\sin x}{x}$.

Solution: Let $f(x) = \sin x$. Then $f(\pi) = \sin 0 = 0$. So

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = \cos(0) = 1.$$

8. Find $\lim_{x\to\pi/4} \frac{\cos x - \sqrt{2}/2}{x - \pi/4}$.

Solution: Let $f(x) = \cos x$. Then $f(\pi/4) = \sqrt{2}/2$. So

$$\lim_{x\to\pi/4} \frac{\cos x - \sqrt{2}/2}{x - \pi/4} = \lim_{x\to\pi/4} \frac{f(x) - f(\pi/4)}{x - \pi/4} = f'(\pi/4) = -\sin \pi/4 = \frac{-\sqrt{2}}{2}.$$

9. Find $\lim_{x\to\pi/3} \frac{\sin x - \sin(\pi/3)}{x - \pi/3}$.

Solution: Let $f(x) = \sin x$. So

$$\lim_{x\to\pi/3} \frac{\sin x - \sin(\pi/3)}{x - \pi/3} = \lim_{x\to\pi/3} \frac{f(x) - f(\pi/3)}{x - \pi/3} = f'(\pi/3) = \cos \pi/3 = \frac{1}{2}.$$

10. Find $\lim_{x\to\pi/3} \frac{\sin x - \sqrt{3}/2}{x - \pi/3}$.

Solution: Let $f(x) = \sin x$. Then $f(\pi/3) = \sqrt{3}/2$. So

$$\lim_{x\to\pi/3} \frac{\sin x - \sqrt{3}/2}{x - \pi/3} = \lim_{x\to\pi/3} \frac{f(x) - f(\pi/3)}{x - \pi/3} = f'(\pi/3) = \cos \pi/3 = \frac{1}{2}.$$